INDIAN STATISTICAL INSTITUTE, Bangalore Centre
Endsem
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Course Name: Complex Analysis Duration: 3 hours

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
Instructions: You are supposed to solve the questions using results proved in class.

1. State whether the following statements are TRUE/ FALSE with proper justification.
   (a) Let $f : \Omega \to \mathbb{C}$ be analytic and $|f(z) - 1| < 1$ for all $z \in \Omega$, then
   $$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$
   for any closed curve γ in Ω. [3]
   (b) Fix $k \geq 1$. Suppose f be an entire function such that the k-th derivative $f^{(k)}$ is polynomial. Then f is necessarily a polynomial. [2]
   (c) Let f be a non-constant entire function, then for every positive real number $r > 0$, the closure of $\{z : |f(z)| < r\}$ is equal to $\{z : |f(z)| \leq r\}$. [3]
   (d) There exists an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(1/2) = 3/4$ and $f'(1/2) = 2/3$. [3]

2. Let $\Omega \subseteq \mathbb{C}$. A real-valued function $u \in C^2(\Omega)$ is called harmonic if
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
   in Ω. Show that real part and imaginary part of any analytic function are harmonic (You may assume without proof that the real and imaginary parts of analytic functions are C^2-smooth).
   [5]

3. [a] Let f be an entire function such that $|f(z)| \leq C|z|^n$ for some $C > 0$ and for all $z \in \mathbb{C}$ with $|z| > R$, for $R > 1$ sufficiently large, then prove that f is a polynomial of degree at most n.
   [b] Suppose that f is bounded and analytic on $\mathbb{C} \setminus \{0\}$. Then prove that f is constant.
   [5+5]

4. (a) Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Re(z)} > 0\}$. Prove that the map
   $$\varphi(z) = \frac{z-1}{z+1}$$
   is a bijective analytic map from $\mathbb{H}$ to $\mathbb{D}$.
   (b) Suppose f is an analytic map from the punctured disc $\mathbb{D} \setminus \{0\}$ such that Re f(z) > 0, then prove that 0 is a removable singularity for f. [3+3]

5. Let f be analytic function from $B(0; R) = z \in \mathbb{C} : 0 \leq |z| < R$ to $\mathbb{C}$. For $0 \leq r < R$, define $M(r) = \max\{\text{Re f(z)} : |\text{z}| = \text{r}\}$. Prove that unless f is identically constant, the function M is strictly increasing. [6]

6. [a] State Rouché's theorem.
   [b] Prove fundamental Theorem of Algebra using Rouché's theorem. [2+5]